A small determinant problem **Yue Kwok Choy**

The terms matrix, determinant and Jacobian, familiar to most mathematics students, was invented by **James Joseph Sylvester (1814-97)**

Question Factorize : $\begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ a^2 & (c+a)^2 & c^2 \\ a^2 & b^2 & (a+b)^2 \end{vmatrix}$

Method 1

$$\begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ a^2 & (c+a)^2 & c^2 \\ a^2 & b^2 & (a+b)^2 \end{vmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ = \\ R_3 \to R_3 - R_1 \end{matrix} \begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ a^2-(b+c)^2 & (c+a)^2-b^2 & 0 \\ a^2-(b+c)^2 & 0 & (a+b)^2-c^2 \end{vmatrix}$$

$$=(a+b+c)^2\begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ a-b-c & c+a-b & 0 \\ a-b-c & 0 & a+b-c \end{vmatrix} \begin{matrix} C_1 \to C_1 - C_2 - C_3 \\ = \end{matrix} (a+b+c)^2\begin{vmatrix} 2bc & b^2 & c^2 \\ -2c & c+a-b & 0 \\ -2b & 0 & a+b-c \end{vmatrix}$$

$$=\frac{2(a+b+c)^2}{bc}\begin{vmatrix} bc & b^2c & bc^2 \\ -c & c(c+a-b) & 0 \\ -b & 0 & b(a+b-c) \end{vmatrix} = 2bc(a+b+c)^2\begin{vmatrix} 1 & b & c \\ -1 & (c+a-b) & 0 \\ -1 & 0 & (a+b-c) \end{vmatrix}$$

$$=2bc(a+b+c)^2\begin{vmatrix} 1 & b & c \\ 0 & c+a & c \\ 0 & b & a+b \end{vmatrix} = 2bc(a+b+c)^2\begin{vmatrix} c+a & c \\ b & a+b \end{vmatrix} = 2bc(a+b+c)^2\left[(a+b)(c+a)-bc\right]$$

$$=\underline{\underline{2abc(a+b+c)^3}}$$

Method 2 Let $f(a, b, c) = \begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ a^2 & (c+a)^2 & c^2 \\ a^2 & b^2 & (a+b)^2 \end{vmatrix}$

Then $f(a, b, c)$ is a homogeneous symmetric expression of a, b, c with degree 6.

$f(0, b, c) = \begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ 0 & c^2 & c^2 \\ 0 & b^2 & b^2 \end{vmatrix} = 0$, since R_2 and R_3 are in ratio .

$\therefore$ abc is a factor of $f(a, b, c)$.

Let $a = -(b + c)$, then $f(-(b + c), b, c) = \begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ (b+c)^2 & b^2 & c^2 \\ (b+c)^2 & b^2 & c^2 \end{vmatrix} = 0$.

$\therefore$ $a + b + c$ is a factor of $f(a, b, c)$.

Similarly, let $b = -(c + a)$ and $c = -(a + b)$, we can get $f = 0$.

$\therefore$ $(a + b + c)^3$ is a factor of $f(a, b, c)$.

$\therefore$ $f(a, b, c) = kabc(a + b + c)^3$

Compare coefficient of a^4bc , we can get $k = 2$. (Only the main diagonal contains the term.)

$\therefore$ $f(a, b, c) = \underline{\underline{2abc(a+b+c)^3}}$